Generalized Toeplitz plus Hankel operators: kernel structure and defect numbers[1]

Victor D. Didenko and Bernd Silbermann

Universiti Brunei Darussalam, Bandar Seri Begawan, BE1410 Brunei; diviol@gmail.com
Technische Universität Chemnitz, Fakultät für Mathematik, 09107 Chemnitz, Germany; silbermn@mathematik.tu-chemnitz.de

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Abstract

Generalized Toeplitz plus Hankel operators $T(a) + H_\alpha(b)$ generated by functions a, b and a linear fractional Carleman shift α changing the orientation of the unit circle $\mathbb{T}$ are considered on the Hardy spaces $H^p(\mathbb{T})$, $1 < p < \infty$. If the functions $a, b \in L^\infty(\mathbb{T})$ and satisfy the condition

$$a(t)a(\alpha(t)) = b(t)b(\alpha(t)), \quad t \in \mathbb{T},$$

the defect numbers of the operators $T(a) + H_\alpha(b)$ are established and an explicit description of the structure of the kernels and cokernels of the operators mentioned is given.

1 Introduction

Let $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$ be the counterclockwise oriented unit circle in the complex plane $\mathbb{C}$. By $PC := PC(\mathbb{T})$ we denote the set of all piecewise continuous function on $\mathbb{T}$, i.e. the set of all functions f such that for any point $t_0 \in \mathbb{T}$ there are finite

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left and right limits $f(t_0-0)$ and $f(t_0+0)$. As usual, $C := C(\mathbb{T})$ refers to the set of all continuous functions on $\mathbb{T}$. Further, let $L^p = L^p(\mathbb{T})$, $1 \le p \le \infty$ stand for the space of all Lebesgue measurable functions f such that

$$\begin{aligned} ||f||_p &:= \left(\int_{\mathbb{T}} |f(t)|^p \, |dt| \right)^{1/p} < \infty, 1 \le p < \infty; \\ ||f||_\infty &:= \operatorname*{ess\,sup}_{t\in\mathbb{T}} |f(t)| < \infty. \end{aligned}$$

If $f \in L^1$, then by $\widehat{f}_n$ we denote the Fourier coefficients of the function f,

$$\widehat{f}_n := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta \,,$$

and for $1 \le p \le \infty$ let $H^p := H^p(\mathbb{T})$ and $\overline{H^p} := \overline{H^p(\mathbb{T})}$ be the Hardy spaces,

$$\begin{aligned} H^p &:= \{ f \in L^p : \widehat{f}_n = 0 \text{ for all } n < 0 \}, \\ \overline{H^p} &:= \{ f \in L^p : \widehat{f}_n = 0 \text{ for all } n > 0 \}. \end{aligned} \tag{1}$$

On the space L^p, $1 < p < \infty$, consider the operators J, P, and Q defined by

$$\begin{aligned} J :& f(t) \to t^{-1} f(t^{-1}) \,, \\ P :& \sum_{n\in\mathbb{Z}} \widehat{f}_n t^n \to \sum_{n\in\mathbb{Z}_+}^{\infty} \widehat{f}_n t^n \,, \\ Q :=& \, I - P \,, \end{aligned} \tag{2}$$

where I is the identity operator, and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ is the set of all non-negative integers. The operators P and Q are complimentary projections and it is well-known that they are bounded on any space L^p for $p \in (1,\infty)$, but not on the spaces L^1 and L^∞.

The classical Toeplitz plus Hankel operators $T(a) + H(b) : H^p \to H^p$, $a, b \in L^\infty$,

$$T(a) + H(b) := PaP + PbQJ \tag{3}$$

are subject to numerous studies, where their Fredholm properties and other related problems have been investigated. Recall that an operator A from the Banach algebra of all linear continuous operators $\mathcal{B}(X)$ acting on a Banach space X is called Fredholm if the range $\operatorname{im} A := \{y \in X : y = Ax, x \in X\}$ of A is a closed subspace of X and the dimensions $\dim \ker A$ and $\dim \operatorname{coker} A$ of the subspaces

$$\ker A := \{x : X : Ax = 0\}, \quad \operatorname{coker} A := \{x \in X : A^* x = 0\}$$

are finite. Here, A^* denotes the adjoint operator to the operator A. As far as the operator $T(a) + H(b)$ is concerned, for $a, b \in PC$ its Fredholm properties can be immediately derived by a direct applications of results [4, Sections 4.95-4.102], [13, Sections 4.5 and 5.7], [14]. The case of quasi piecewise continuous generating functions has been studied in [16], whereas formulas for the index of the operators (3) considered on various Banach and Hilbert spaces and with various assumptions about the generating functions a and b have been established in [5, 15]. Recently, progress has been made in computation of defect numbers $\dim\ker(T(a)+H(b))$ and $\dim\operatorname{coker}(T(a) + H(b))$ for various classes of generating functions a and b [3, 7]. Moreover, a more delicate problem of the description of the spaces $\ker(T(a)+H(b))$ and $\operatorname{coker}(T(a) + H(b))$ has been considered [7, 8].

The aim of the present work is to study generalized Toeplitz plus Hankel operators. These operators are similar to the classical Toeplitz plus Hankel operator (2) but the flip operator J of (3) is replaced by another operator J_α generated by a linear fractional shift α changing the orientation of the circle $\mathbb{T}$. Areas of particular interest to us are the kernels and cokernels of such operators and we are going to derive an explicit description of these spaces in the case where the generating functions a and b belong to the space L^∞ and satisfy an additional algebraic relation. Note that generalized Toeplitz plus Hankel operators have been previously considered in [10, 11] but under more restrictive assumptions. Moreover, in the present work we single out certain classes of generalized Toeplitz plus Hankel operators which are subject to Coburn–Simonenko theorem. Recall that the classical Coburn–Simonenko Theorem claims that if a is a non-zero function, then at least one of the numbers $\dim\ker T(a)$ or $\dim\operatorname{coker} T(a)$ is equal to zero. For a Fredholm Toeplitz operator $T(a)$, the generating function a is invertible. Therefore, the Coburn–Simonenko Theorem indicates that any Fredholm Toeplitz operator is one-sided invertible.

Let β be a complex number such that $|\beta| > 1$, and let $\mathbb{S}^2$ denote the Riemann sphere. Consider the mapping $\alpha : \mathbb{S}^2 \to \mathbb{S}^2$ defined by

$$\alpha(z) := \frac{z-\beta}{\overline{\beta} z - 1}, \tag{4}$$

and recall some basic properties of α. In particular, one has:

(i) The mapping $\alpha : \mathbb{S}^2 \to \mathbb{S}^2$ is an one-to-one, $\alpha(\mathbb{T}) = \mathbb{T}$, and if $D^+ := \{z \in \mathbb{C} : |z| < 1\}$ is the interior of the unite circle $\mathbb{T}$ and $\overline{D^+} := D^+ \cup \mathbb{T}$, then

$$\begin{aligned} \alpha(D^+) &= \mathbb{S}^2 \setminus \overline{D^+}, \\ \alpha(\mathbb{S}^2 \setminus \overline{D^+}) &= D^+. \end{aligned} \tag{5}$$

It is clear that α is an automorphism of the Riemann sphere. and the mappings $H^p \to \overline{H^p}$, $h \mapsto h \circ \alpha$ and $\overline{H^p} \to H^p$, $h \mapsto h \circ \alpha$ are well-defined isomorphisms.

Note that in the last relations, the mapping α is understood as acting on the unit circle $\mathbb{T}$. A proof of this result can be given by using relations (10). We omit the details here but mention that they can be found in the proof of Proposition 2.2 of [11].

(ii) The mapping $\alpha : \mathbb{T} \to \mathbb{T}$ changes the orientation of $\mathbb{T}$, satisfies the Carleman condition $\alpha(\alpha(t)) = t$ for all $t \in \mathbb{T}$, and possesses two fixed points $t_+ = (1+\lambda)/\overline{\beta}$ and $t_- = (1-\lambda)/\overline{\beta}$, where $\lambda := i\sqrt{|\beta|^2 - 1}$.

(iii) The mapping α admits the factorization

$$\alpha(t) = \alpha_+(t)\, t^{-1}\, \alpha_-(t), \tag{6}$$

where

$$\alpha_+(t) = \frac{t-\beta}{\lambda}, \quad \alpha_-(t) = \frac{\lambda t}{\overline{\beta} t - 1}, \tag{7}$$

and $\alpha_+^{\pm 1} \in H^\infty$, $\alpha_-^{\pm 1} \in \overline{H^\infty}$.

(iv) On the space L^p, $1 < p < \infty$, the mapping α generates a bounded linear operator J_α, called weighted shift operator and defined by

$$J_\alpha \varphi(t) := t^{-1} \alpha_-(t) \varphi(\alpha(t)), \quad t \in \mathbb{T}. \tag{8}$$

Further, for $a \in L^\infty$ let a_α denote the composition of the functions a and α, i.e.

$$a_\alpha(t) := a(\alpha(t)), \quad t \in \mathbb{T}.$$

It is clear that for any $a \in L^\infty$, the operator J_α satisfies the relation

$$J_\alpha a I = a_\alpha J_\alpha, \tag{9}$$

and for any $n \in \mathbb{Z}$, one has $(a^n)_\alpha = (a_\alpha)^n := a_\alpha^n$.

In addition,

$$J_\alpha^2 = I, \quad J_\alpha P = Q J_\alpha, \quad J_\alpha Q = P J_\alpha, \tag{10}$$

and

$$\alpha_+^{\pm 1}(\alpha(t)) = \alpha_-^{\pm 1}(t), \quad \overline{\alpha}_+^{\pm 1}(t) = \alpha_-^{\mp 1}(t). \tag{11}$$

Indeed, consider for example the second identity in (11). Then

$$\overline{\alpha_+(t)} = \overline{\left(\frac{t-\beta}{\lambda}\right)} = \frac{1/t - \overline{\beta}}{-\lambda} = \frac{\overline{\beta} t - 1}{t\lambda} = \alpha_-^{-1}(t).$$

Let us briefly discuss the situation with adjoint operators. Recall that the adjoint to H^p is the space $L^q/\operatorname{im} Q$, $p^{-1}+q^{-1}=1$. It can be identified with the space H^q but equipped with an equivalent norm coinciding with usual one in the case $p=2$ (see [4, Section 1.42]). For our purposes here, there is no need to distinguish between the two spaces H^q and $L^q/\operatorname{im} Q$, and we can also assume that the adjoint operator A^* to the operator $A \in \mathcal{L}(H^P)$ is acting on the space H^q.

On the space H^p, $1<p<\infty$, any element $a \in L^\infty$ defines two operators $T(a)$ and $H_\alpha(a)$ such that

$$\begin{aligned} T(a) &: \varphi \mapsto Pa\varphi, \\ H_\alpha(a) &: \varphi \mapsto PbQJ_\alpha\varphi. \end{aligned}$$

The operator $T(a)$ is referred to as Toeplitz operator generated by the function a, whereas the operator $H_\alpha(a)$ is called generalized Hankel operator generated by the function a and the shift α. Many properties of generalized Hankel operators are similar to the corresponding properties of classical Hankel operators $PbQJ$. For example, Toeplitz and generalized Hankel operators are connected in the following way,

$$\begin{aligned} T(cd) &= T(c)T(d) + H_\alpha(c)H_\alpha(d_\alpha), \\ H_\alpha(cd) &= T(c)H_\alpha(d) + H_\alpha(c)T(d_\alpha). \end{aligned} \tag{12}$$

For classical Hankel operators such kind formulas are well known [4], and they are often used in various studies of Toeplitz and Hankel operators.

Let $a, b \in L^\infty$. In the present paper we study the properties of the operators $T(a)+H_\alpha(b) : H^p \to H^p$, $1<p<\infty$. Such operators are called generalized Toeplitz plus Hankel operators generated by the functions a, b and by the shift α or simply generalized Toeplitz plus Hankel operators. Similarly, $T(a)-H_\alpha(b)$ are called generalized Toeplitz minus Hankel operators.

Recall that the kernel and cokernel dimensions of generalized Toeplitz plus Hankel operators and even more general operators have been studied in [10, 11]. The approach of [10, 11] is based on a special factorization of the operators in question, which in turn involves factorizations of matrix functions. Thus assuming that the corresponding operators are Fredholm, the authors express the kernel and cokernel dimensions in terms of the partial indices of some matrix valued functions. However, it is well known that for an arbitrary matrix function the computation of its partial indices is a very demanding task. Moreover, in the most cases this is not possible at all. Therefore, an important problem is to find certain classes of operators where more efficient results can be derived. One such a class of generalized Toeplitz plus Hankel operators $T(a)+H_\alpha(b)$, $a, b \in L^\infty$ is mentioned in [11]. It can be characterized by the relation

$$aa_\alpha = bb_\alpha,$$

and in [11] such operators are studied under the following assumptions.

(i) The operators $T(a) + H_\alpha(b) : H^p \to H^p$ and $T(a) - H_\alpha(b) : H^p \to H^p$ are Fredholm.

(ii) The functions a and ba^{-1} admit *bounded* Wiener–Hopf factorizations.

In the present work, the conditions imposed on the operator $T(a) + H_\alpha(b)$ are more general and a completely different approach to these operators is used. In particular, condition i) is replaced by a weaker one and we also avoid *bounded* Wiener–Hopf factorization. Recall that a Toeplitz operator $T(a)$ acting on H^p is Fredholm if and only if the generating function a admits a Wiener–Hopf factorization in the sense of the definition given in Section 4. The reader can also find there a brief discussion of the two types of Wiener–Hopf factorizations mentioned.

This paper is organized as follows. In Section 2, connections between the kernels of the operators $T(a) \pm H_\alpha(b)$ and the kernel of a matrix Toeplitz operator $T(V(a,b))$ are considered. Here we also describe the structure of $\ker T(V(a,b))$ in terms of the kernels of certain scalar Toeplitz operators. This allows us to establish a general representation for the kernels of the operators $T(a) \pm H_\alpha(b)$. In Section 3, some classes of generalized Toeplitz plus Hankel operators, where Coburn–Simonenko theorem holds, are studied. For classical Toeplitz plus Hankel operators similar problems are discussed in [3, 7, 8], whereas [6] deals with Wiener–Hopf plus Hankel operators. In Section 4, a decomposition for the kernels of special scalar Toeplitz operators is derived. This decomposition is employed in Section 5 in order to give a complete and effective description for the kernels and cokernels of the generalized Toeplitz plus Hankel operators under consideration. The remaining part of the paper is devoted to the operators with piecewise continuous generating functions a and b.

2 Kernels of generalized Toeplitz plus Hankel operators and related matrix Toeplitz operators.

There are well known relations between classical Toeplitz plus Hankel operators and matrix Toeplitz operators. Similar formulas can be also derived for the operators $T(a) + H_\alpha(b)$. Moreover, these formulas can be effectively used in order to establish connections between the kernels of the corresponding operators and to derive a complete and efficient description of the kernels of the operators under consideration. More precisely, let $a, b \in L^\infty$ and $a \in GL^\infty$, where GL^∞ denotes the group of invertible elements in L^∞. The last assumption about the generating function a is justified by the fact that the semi-Fredholmness of the operator $T(a) + H_\alpha(b)$ implies that $a \in GL^\infty$. Along with $T(a) + H_\alpha(b)$ consider the generalized Toeplitz minus

Hankel operators $T(a) - H_\alpha(b)$ and matrix Toeplitz operator $T(V(a,b))$, where

$$V(a,b) := \begin{pmatrix} a - bb_\alpha a_\alpha^{-1} & d \\ -c & a_\alpha^{-1} \end{pmatrix}, \quad c := b_\alpha a_\alpha^{-1},\ d := ba_\alpha^{-1}.$$

The following result plays an important role in the description of the kernels of Toeplitz plus Hankel operators.

Lemma 2.1 *Assume that $a, b \in L^\infty$, $a \in GL^\infty$, and the operators $T(a) \pm H_\alpha(b)$ are considered on the space H^p, $1 < p < \infty$. Then*

- *If $(\varphi, \psi)^T \in \ker T(V(a,b))$, then*

$$\begin{aligned}(\Phi, \Psi)^T =& \frac{1}{2}(\varphi - J_\alpha Qc\varphi + J_\alpha Q a_\alpha^{-1}\psi, \varphi + J_\alpha Qc\varphi - J_\alpha Q a_\alpha^{-1}\psi)^T \\ &\in \ker \operatorname{diag}(T(a) + H_\alpha(b), T(a) - H_\alpha(b))\end{aligned} \tag{13}$$

- *If $(\Phi, \Psi)^T \in \ker \operatorname{diag}(T(a) + H_\alpha(b), T(a) - H_\alpha(b))$, then*

$$(\Phi + \Psi, P(b_\alpha(\Phi + \Psi) + a_\alpha J_\alpha P(\Phi - \Psi))^T \in \ker T(V(a,b)). \tag{14}$$

Moreover, the operators

$$\begin{aligned}&\mathbf{U}_1 : \ker T(V(a,b)) \to \ker \operatorname{diag}(T(a) + H_\alpha(b), T(a) - H_\alpha(b)),\\ &\mathbf{U}_2 : \ker \operatorname{diag}(T(a) + H_\alpha(b), T(a) - H_\alpha(b)) \to \ker T(V(a,b)),\end{aligned}$$

defined, respectively, by the relations (13) *and* (14) *are mutually inverses to each other.*

Proof. First all all we note that the operator $T(a) + H_\alpha(b) \in \mathcal{L}(H^p(\mathbb{T}))$ and $(T(a) + H_\alpha(b))P + Q \in \mathcal{L}(L^p(\mathbb{T}))$ are simultaneously invertible or Fredholm and have the same index. For simplicity, in what follows we will write $T(a) \pm H_\alpha(b) + Q$ instead of $(T(a) \pm H_\alpha(b))P + Q$. On the space $L^p \times L^p$ consider the operator $\operatorname{diag}(T(a) + H_\alpha(b) + Q, T(a) - H_\alpha(b) + Q)$ and represent it as the product of three matrix operators, viz.

$$\begin{pmatrix} T(a) + H_\alpha(b) + Q & 0 \\ 0 & T(a) - H_\alpha(b) + Q \end{pmatrix} = B(T(V(a,b)) + \operatorname{diag}(Q,Q))A\,, \tag{15}$$

where $A, B : L^p \times L^p \to L^p \times L^p$ are invertible operators,

$$A = \begin{pmatrix} I & 0 \\ b_\alpha I & a_\alpha I \end{pmatrix} \begin{pmatrix} I & I \\ J_\alpha & -J_\alpha \end{pmatrix}, \tag{16}$$

and $B = B_1 B_2 B_3$ with

$$B_1 = 2\begin{pmatrix} I & J_\alpha \\ I & -J_\alpha \end{pmatrix},$$

$$B_2 = \mathrm{diag}\,(I, I) - \mathrm{diag}\,(P, Q)\begin{pmatrix} aI & bI \\ b_\alpha I & a_\alpha I \end{pmatrix}\mathrm{diag}\,(Q, P),$$

$$B_3 = \mathrm{diag}\,(I, I) + \mathrm{diag}\,(P, P)\begin{pmatrix} a - bb_\alpha a_\alpha^{-1} & dI \\ -cI & a_\alpha^{-1} I \end{pmatrix}\mathrm{diag}\,(Q, Q).$$

The representation (15) can be verified by straightforward computations using relations (9)-(10). The rest of the proof goes through as for Lemma 3.2 of [8]. ∎

Remark 2.1 *Relation* (15) *shows that the matrix operators* $\mathrm{diag}\,(T(a) + H_\alpha(b), T(a) - H_\alpha(b))$ *and* $T(V(a,b))$ *are simultaneously Fredholm. This is not always true for the operators* $T(a) + H_\alpha(b)$ *and* $T(a) - H_\alpha(b)$ *themselves. Moreover, even if both operators* $T(a) + H_\alpha(b)$ *and* $T(a) - H_\alpha(b)$ *are Fredholm, they can have different indices. Nevertheless, the relation* (15) *plays an important role in what follows and can be employed in order to obtain a description of the kernels of generalized Toeplitz plus Hankel operators.*

From now on we will always assume that a belongs to the group GL^∞ of invertible elements from L^∞ and that the generating functions a and b satisfy the condition

$$aa_\alpha = bb_\alpha. \tag{17}$$

Relation (17) is called matching condition, whereas the corresponding duo (a, b) is referred to as an α-matching pair or simply a matching pair. For each α-matching pair (a, b) one can assign another α-matching pair (c, d), where $c = ab^{-1}$ and $d = ba_\alpha^{-1}$. Such a pair (c, d) is called the subordinated pair for (a, b), and it is easily seen that the functions which constitutes a subordinated pair have a specific property, namely $cc_\alpha = 1 = dd_\alpha$. Throughout this paper any function $g \in L^\infty$ satisfying the condition

$$gg_\alpha = 1$$

is called α-matching or simply matching function. In passing note that for the subordinated pair (c, d), functions c and d can also be expressed in the form

$$c = b_\alpha a_\alpha^{-1}, \quad d = b_\alpha^{-1} a.$$

Besides, if (c, d) is the subordinated pair for an α-matching pair (a, b), then $(\overline{d}, \overline{c})$ is the subordinated pair for the matching pair $(\overline{a}, \overline{b}_\alpha)$ defining the adjoint operator

$$(T(a) + H_\alpha(b))^* = T(\overline{a}) + H_\alpha(\overline{b}_\alpha), \tag{18}$$

to the operator $T(a) + H_\alpha(b)$. Note that in order to derive formula (18), one can use the relation

$$J_\alpha^*\psi(t) = t^{-1}\alpha_-(t)\psi(\alpha(t)), \quad t \in \mathbb{T}. \tag{19}$$

Further, a matching pair (a, b) is called Fredholm, if the Toeplitz operators $T(c)$ and $T(d)$ are Fredholm.

Henceforth the operator aI of multiplication by the function $a \in L^\infty$ is denoted simply by a. Note that if (a, b) is a matching pair, then the corresponding matrix–function $V(a, b)$ takes the form

$$V(a, b) = \begin{pmatrix} 0 & d \\ -c & a_\alpha^{-1} \end{pmatrix}.$$

where (c, d) is the subordinated pair for the pair (a, b). In addition, by [8] the operator $T(V(a, b))$ can be represented as the product of three matrix operators, namely,

$$\begin{aligned} T(V(a, b)) &= \begin{pmatrix} 0 & T(d) \\ -T(c) & T(a_\alpha^{-1}) \end{pmatrix} \\ &= \begin{pmatrix} -T(d) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & T(a_\alpha^{-1}) \end{pmatrix} \begin{pmatrix} -T(c) & 0 \\ 0 & I \end{pmatrix}, \end{aligned} \tag{20}$$

where the operator

$$D := \begin{pmatrix} 0 & -I \\ I & T(a_\alpha^{-1}) \end{pmatrix}$$

in the right–hand side of (20) is invertible and

$$D^{-1} = \begin{pmatrix} T(a_\alpha^{-1}) & I \\ -I & 0 \end{pmatrix}.$$

Further, let us also mention a useful representation for the kernel of the block Toeplitz operator $T(V(a, b))$ established recently in the context of classical Toeplitz plus Hankel operators. Thus the following result holds.

Proposition 2.1 ([8]) *Let $(a, b) \in L^\infty \times L^\infty$ be a matching pair such that the operator $T(c)$, $c = ab^{-1}$, is invertible from the right. Then*

$$\ker T(V(a, b)) = \Omega(c) \dotplus \widehat{\Omega}(d)$$

where

$$\begin{aligned} \Omega(c) &:= \left\{(\varphi, 0)^T : \varphi \in \ker T(c)\right\}, \\ \widehat{\Omega}(d) &:= \left\{(T_r^{-1}(c)T(a_\alpha^{-1})s, s)^T : s \in \ker T(d)\right\}, \end{aligned}$$

and $T_r^{-1}(c)$ is one of the right inverses for the operator $T(c)$.

Thus the inclusion $\varphi \in \ker T(c)$ implies that $(\varphi, 0)^T \in \ker T(V(a, b))$ and by Lemma 2.1 one obtains

$$\begin{aligned} \varphi - J_\alpha Q c P \varphi &\in \ker(T(a) + H_\alpha(b)), \\ \varphi + J_\alpha Q c P \varphi &\in \ker(T(a) - H_\alpha(b)). \end{aligned} \tag{21}$$

It is even more remarkable that the functions $\varphi - J_\alpha QcP\varphi$ and $\varphi + J_\alpha QcP\varphi$ belong to the kernel of the operator $T(c)$ as well.

Proposition 2.2 *Assume that $g \in L^\infty$ is a matching function and $f \in \ker T(g)$. Then $J_\alpha QgPf \in \ker T(g)$ and $(J_\alpha QgP)^2 f = f$.*

Proof. If $gg_\alpha = 1$ and $f \in \ker T(g)$, then

$$\begin{aligned} T(g)(J_\alpha QgPf) &= PgPJ_\alpha QgPf = J_\alpha Qg_\alpha QgPf \\ &= J_\alpha Qg_\alpha gPf - J_\alpha Qg_\alpha PgPf = 0, \end{aligned}$$

i.e. $J_\alpha QgPf \in \ker T(g)$. On the other hand, for any $f \in \ker T(g)$ one has

$$\begin{aligned} (J_\alpha QgP)^2 f &= J_\alpha QgPJ_\alpha QgPf = Pg_\alpha QgPf \\ &= Pg_\alpha gPf - Pg_\alpha PgPf = f - Pg_\alpha T(g)f = f, \end{aligned}$$

and we are done. ■

Let us introduce the operator $\mathbf{P}_\alpha(g) := J_\alpha QgP\big|_{\ker T(g)}$. Proposition 2.2 implies that $\mathbf{P}_\alpha(g) : \ker T(g) \to \ker T(g)$ and $\mathbf{P}_\alpha^2(g) = I$. Thus on the space $\ker T(g)$ the operators $\mathbf{P}_\alpha^\pm(g) := (1/2)(I \pm \mathbf{P}_\alpha(g))$ are complementary projections, so they generate a decomposition of $\ker T(g)$. Moreover, using (21) one can formulate the following result.

Corollary 2.1 *Let (c, d) be the subordinated pair for the matching pair $(a, b) \in L^\infty \times L^\infty$. Then*

$$\ker T(c) = \operatorname{im} \mathbf{P}_\alpha^-(c) \dotplus \operatorname{im} \mathbf{P}_\alpha^+(c),$$

and

$$\begin{aligned} \operatorname{im} \mathbf{P}_\alpha^-(c) &\subset \ker(T(a) + H_\alpha(b)), \\ \operatorname{im} \mathbf{P}_\alpha^+(c) &\subset \ker(T(a) - H_\alpha(b)), \end{aligned} \tag{22}$$

hold.

Relations (22) show the influence of the operator $T(c)$ on the kernels of the operators $T(a)+H_\alpha(b)$ and $T(a)-H_\alpha(b)$. Let us now clarify the role of the other operator–viz. the operator $T(d)$, in the description of the kernels of the operators $T(a) + H_\alpha(b)$ and $T(a) - H_\alpha(b)$. Assume additionally that the operator $T(c)$ is invertible from

the right. If $s \in \ker T(d)$, then the element $(T_r^{-1}(c)T(a_\alpha^{-1})s, s)^T \in \ker T(V(a,b))$. By Lemma 2.1, the element

$$2\varphi_\alpha^\pm(s) := T_r^{-1}(c)T(a_\alpha^{-1})s \mp J_\alpha QcPT_r^{-1}(c)T(a_\alpha^{-1})s \pm J_\alpha Qa_\alpha^{-1}s \tag{23}$$

belongs to the null space $\ker(T(a) \pm H_\alpha(b))$ of the corresponding operator $T(a) \pm H_\alpha(b)$.

Lemma 2.2 *The mapping $s \mapsto \varphi_\alpha^\pm(s)$ is a one-to-one function from the space $\operatorname{im} \mathbf{P}_\alpha^\pm(d)$ to the space $\ker(T(a) \pm H_\alpha(b))$.*

Proof. Assuming that s belongs to the kernel of the operator $T(d)$, one can show that the operator $(1/2)(Pb_\alpha P + Pa_\alpha J_\alpha P)$ sends the element $\varphi_\alpha^+(s)$ into $\mathbf{P}_\alpha^+(d)s$ and the operator $(1/2)(Pb_\alpha P - Pa_\alpha J_\alpha P)$ sends the element $\varphi_\alpha^-(s)$ into $\mathbf{P}_\alpha^-(d)s$. The proof of these facts is based on the relations (10) and proceeds similarly to the proof of [8, Lemma 3.6]. Then Lemma 2.2 follows. ∎

Proposition 2.3 *Let (c,d) be the subordinated pair for a matching pair $(a,b) \in L^\infty \times L^\infty$. If the operator $T(c)$ is right-invertible, then*

$$\begin{aligned} \ker(T(a) + H_\alpha(b)) &= \varphi_\alpha^+(\operatorname{im} \mathbf{P}_\alpha^+(d)) \dotplus \operatorname{im} \mathbf{P}_\alpha^-(c), \\ \ker(T(a) - H_\alpha(b)) &= \varphi_\alpha^-(\operatorname{im} \mathbf{P}_\alpha^-(d)) \dotplus \operatorname{im} \mathbf{P}_\alpha^+(c). \end{aligned} \tag{24}$$

Moreover,

$$\varphi_\alpha^+(\operatorname{im} \mathbf{P}_\alpha^-(d)) \subset \operatorname{im} \mathbf{P}_\alpha^-(c), \quad \varphi_\alpha^-(\operatorname{im} \mathbf{P}_\alpha^+(d)) \subset \operatorname{im} \mathbf{P}_\alpha^+(c).$$

The proof of this proposition is similar to the proof of the corresponding results of [8] for classical Toepltz plus Hankel operators.

3 Some classes of generalized Toeplitz plus Hankel operators and Coburn–Simonenko theorem.

The aim of this section is to show how results of the previous section can be exploited in order to study generalized Toeplitz plus Hankel operators with generating functions a and b connected in a special way. Similar classical Toeplitz plus Hankel operators have been studied in [1, 2, 9]. Nevertheless our approach seems to be more simple and allows us to treat the operators concerned from a unified point of view. In particular, as it will be shown below, the operators in question satisfy the Coburn–Simonenko theorem. Recall that this theorem states that if a is a non-zero function, then either kernel or cokernel of a scalar Toeplitz operator $T(g)$, $g \in L^\infty$ is trivial. In general, generalized Toeplitz plus Hankel operators do not possess such a

property. However, for some classes of operators $T(a)+H_\alpha(b)$ a version of Coburn–Simonenko theorem still holds. It is worth noting that the approach here does not use factorization technique.

Let us write the operator $J_\alpha : L^p \mapsto L^p$ as follows

$$J_\alpha\varphi(t) := \chi^{-1}(t)\varphi(\alpha(t)),$$

where

$$\chi(t) = \frac{t}{\alpha_-(t)}, \quad t \in \mathbb{T}.$$

The function $\chi \in H^\infty$ and has a number of remarkable properties. Some of them are listed in the lemma below.

Lemma 3.1 *Let the shift α be as above. Then*

(i) The function χ is a matching function, i.e. $\chi\chi_\alpha = 1$, and $\operatorname{wind}\chi = 1$, *where* $\operatorname{wind}\chi$ *denotes the winding number of the function χ.*

(ii) The function $\chi_\alpha \in \overline{H^\infty}$ and $\chi_\alpha(\infty) = 0$.

(iii) If $a, b \in L^\infty$ and n is a positive integer, then

$$T(a) + H_\alpha(b) = (T(a\chi^{-n}) + H_\alpha(b\chi^n))T(\chi^n). \tag{25}$$

Proof. The proof of the first item is a matter of straightforward computation, whereas the inclusion $\chi_\alpha \in \overline{H^\infty}$ follows from (7). The identity (25) is a consequence of relations (12) and the fact that $H_\alpha(\chi^n)T(\chi^n) = 0$ for any $n \in \mathbb{N}$. ■

Now we can establish the main result of this section.

Theorem 3.1 *Let $a \in GL^\infty$, and let A be any of the operators $T(a) - H_\alpha(a\chi^{-1})$, $T(a) + H_\alpha(a\chi)$, $T(a) + H_\alpha(a)$, $T(a) - H_\alpha(a)$. Then* $\ker A = \{0\}$ *or* $\operatorname{coker} A = \{0\}$.

Proof. Consider first the operator $T(a)+H_\alpha(a\chi)$. The duo $(a, a\chi)$ is a matching pair with the subordinated pair (c, d), where $c = \chi^{-1}$ and $d = aa_\alpha^{-1}\chi$. Note that $\operatorname{wind} c = -1$ so that the operator $T(c) = T(\chi^{-1})$ is invertible from the right and $\dim\ker T(\chi^{-1}) = 1$. Moreover, it is easily seen that the constant function $\mathbf{e} := \mathbf{e}(t) = 1$, $t \in \mathbb{T}$, belongs both to $\ker T(\chi^{-1})$ and $\ker(T(a) - H_\alpha(a\chi))$. Thus $\ker T(\chi^{-1}) = \{\lambda\mathbf{e}\} = \operatorname{im}\mathbf{P}_\alpha^-(\chi^{-1}) \dotplus \operatorname{im}\mathbf{P}_\alpha^+(\chi^{-1})$, $\lambda \in \mathbb{C}$. Besides

$$\begin{aligned}\mathbf{P}_\alpha^-(\chi^{-1})\mathbf{e} &= (1/2)(\mathbf{e} - J_\alpha Q\chi^{-1}P\mathbf{e})\\ &= (1/2)(\mathbf{e} - J_\alpha(\chi^{-1}\mathbf{e})) = (1/2)(\mathbf{e} - \chi_\alpha^{-1}\chi^{-1}\mathbf{e}) = 0,\end{aligned}$$

so $\operatorname{im}\mathbf{P}_\alpha^-(\chi^{-1}) = \{0\}$ and Proposition 2.3 implies that

$$\dim\ker(T(a) + H_\alpha(a\chi)) = \dim\operatorname{im}\mathbf{P}_\alpha^+(d). \tag{26}$$

with $d = aa_\alpha^{-1}\chi$. Let $\dim\ker T(d) > 0$. Then the Coburn–Simonenko theorem gives that

$$\operatorname{coker} T(\chi^{-1}) = \operatorname{coker} T(d) = \{0\}.$$

Factorization (20) entails that the cokernel of $T(V(a, a\chi))$ is trivial. By (15) the cokernel of the diagonal operator $\operatorname{diag}(T(a)+H_\alpha(a\chi), T(a)-H_\alpha(a\chi))$ is also trivial, and hence so is the cokernel of $T(a)+H_\alpha(a\chi)$ and that of $T(a)-H_\alpha(a\chi)$. On the other hand, if $\dim\ker T(d) = 0$, then (26) leads to the conclusion that

$$\ker(T(a) + H_\alpha(a\chi)) = \{0\},$$

so the operator $T(a) + H_\alpha(a\chi)$ is subject to the Coburn–Simonenko theorem.

Consider now the operator $T(a) - H_\alpha(\chi^{-1}a)$. The representation (25) implies that

$$T(a) - H_\alpha(a\chi^{-1}) = (T(a\chi^{-1}) - H_\alpha((a\chi^{-1})\chi)) \cdot T(\chi). \tag{27}$$

Setting $b := a\chi^{-1}$, one rewrites the first operator in the right-hand side of (27) as

$$T(a\chi^{-1}) - H_\alpha((a\chi^{-1})\chi) = T(b) - H_\alpha(b\chi).$$

The operators of the form $T(b) - H_\alpha(b\chi)$ have been already considered, and it was mentioned that the element $\mathbf{e}$ belongs to the kernel of the operator $T(b) - H_\alpha(b\chi)$. Further, it is not difficult to see that $\mathbf{e} \notin \operatorname{im} T(\chi) = \operatorname{im} T(\overline{\beta}t - 1)$. Indeed, if $\mathbf{e} \in \operatorname{im} T(\overline{\beta}t - 1)$, then there is a function $h \in H^p$ such that

$$1 = (\overline{\beta}t - 1)h. \tag{28}$$

However, the function $\overline{\beta}t - 1$ vanishes at the point $t = 1/\overline{\beta} \in D^+$. The function h admits an analytical extension into domain D^+, whereas relation (28) is still valid for the domain D^+. But this is a contradiction. Write $d = bb_\alpha^{-1}\chi$ and assume that $\ker T(d) = \{0\}$. Since $\mathbf{e} \notin \operatorname{im} T(\chi)$, relation (27) implies that

$$\ker(T(a) - H_\alpha(a\chi^{-1})) = \{0\}.$$

On the other hand, if $\dim\ker T(d) > 0$, then

$$\ker(T(b) - H_\alpha((b\chi)) = \varphi_\alpha^-(\operatorname{im} \mathbf{P}_\alpha^-(d)) + \operatorname{span}\{\mathbf{e}\}$$

and

$$\operatorname{coker}(T(b) - H_\alpha(b\chi)) = \{0\}.$$

Since $\operatorname{ind} T(\overline{\beta}t - 1) = -1$, we have that $\operatorname{im} T(\overline{\beta}t - 1) \dotplus \{\lambda\mathbf{e}\} = H^p$, $\lambda \in \mathbb{C}$. Let K denote the projection which maps the space H^p onto $\operatorname{im} T(\overline{\beta}t - 1)$ in parallel to $\{\lambda\mathbf{e}\}$. Each element $s \in \ker(T(b) - H_\alpha(b\chi))$ can be represented as

$$s = Ks + (I - K)s,$$

and $(I-K)s$ is a constant. Thus $(I-K)s \in \ker(T(b) - H_\alpha(b\chi))$ and this implies that $Ks \in \ker(T(b) - H_\alpha(b\chi))$. Using (27), we get that

$$\operatorname{coker}(T(a) - H_\alpha(a\chi^{-1})) = 0.$$

The remaining operators $T(a) + H_\alpha(a)$ and $T(a) - H_\alpha(a)$ can be considered analogously. ■

Before concluding this section, let us mention a certain duality for the operators in Theorem 3.1. Along with $\chi(t) := t/\alpha_-(t)$ consider the function $\Psi(t) = t/\alpha_+(t)$. Then the following result is true.

Corollary 3.1 *Let $a \in GL^\infty$, and let A be any of the operators $T(a) - H_\alpha(a_\alpha \Psi^{-1})$, $T(a) + H_\alpha(a_\alpha \Psi)$, $T(a) + H_\alpha(a_\alpha)$, $T(a) - H_\alpha(a_\alpha)$. Then $\ker A = 0$ or $\operatorname{coker} A = 0$.*

The proof of this corollary directly follows from Theorem 3.1 and the representation (18) for the adjoint of generalized Toeplitz plus Hankel operator.

4 A kernel decomposition for Toeplitz operators with α-matching generating functions.

In this section we study the kernels of the Toeplitz operators $T(g)$ in the case where the generating function $g \in L^\infty$ is an α-matching function, i.e. if it satisfies the relation $gg_\alpha = 1$. Thus we describe bases in the spaces $\operatorname{im} \mathbf{P}^\pm_\alpha(g)$. These results lay the foundation for the basis construction of the kernel of generalized Toeplitz plus Hankel operators with generating matching functions.

Let us start by recalling some properties of Toeplitz operators. It is well known that Fredholmness of the operator $T(a)$ is closely connected to Wiener–Hopf factorization of the corresponding generating function a. Let $p > 1$, $q > 1$ be real numbers such that $p^{-1} + q^{-1} = 1$.

4.1 *A function $g \in L^\infty$ admits a week Wiener–Hopf factorization in H^p, if it can be represented in the form*

$$g = g_- t^n g_+, \quad g_-(\infty) = 1, \tag{29}$$

where $n \in \mathbb{Z}$, $g_+ \in H^q$, $g_+^{-1} \in H^p$, $g_- \in \overline{H^p}$, $g_-^{-1} \in \overline{H^q}$.

The weak Wiener–Hopf factorization of a function g is unique, if it exists. The functions g_- and g_+ are called the factorization factors, and the number n is the factorization index. If $g \in L^\infty$ and the operator $T(g)$ is Fredholm, the function g admits the weak Wiener–Hopf factorization with an index $n = -\operatorname{ind} T(g)$ [4, 12]. Besides, in this case, the factorization factors possess an additional property–viz.

the linear operator $g_+^{-1}Pg_-^{-1}I$ defined on span $\{t^k : k \in \mathbb{Z}_+\}$ can be boundedly extended on the whole space H^p. In the following, such a kind of weak Wiener–Hopf factorization in H^p is called simply Wiener–Hopf factorization in H^p.

A Wiener–Hopf factorization is called *bounded* if the factorization factors g_+, g_+^{-1} and g_-, g_-^{-1} belong to H^∞ and $\overline{H^\infty}$, respectively. Obviously, a bounded weak Wiener–Hopf factorization is automatically a Wiener–Hopf factorization which does not depend on p. However, it is worth mentioning that there are continuous non-vanishing on $\mathbb{T}$ functions which do not admit bounded Wiener–Hopf factorization. Moreover, let us also recall that if $h_1 \in H^\infty$, $h_2 \in \overline{H^\infty}$ and $g \in L^\infty$, then $T(h_2 g h_1) = T(h_2)T(g)T(h_1)$. The last relation is an immediate consequence of the already mentioned formula $T(ab) = T(a)T(b) + H(a)H(\widetilde{b})$, where $\widetilde{b} := b(1/t)$.

Theorem 4.1 (see **[4, Section 5.5]**) *If $g \in L^\infty$, then the Toeplitz operator $T(g) : H^p \to H^p$, $1 < p < \infty$ is Fredholm and* $\operatorname{ind} T(g) = -n$ *if and only if the generating function g admits the Wiener–Hopf factorization* (29) *in H^p.*

Let us emphasize that Fredholmness of a Toeplitz operator depends on the space where this operator acts (see [4, 12]) and in many cases there are efficient formulas to compute the index of the operator $T(g)$ and, therefore, its defect numbers $\dim\ker T(g)$ and $\dim\operatorname{coker} T(g)$ due to the Coburn–Simonenko Theorem. We also recall that one-sided inverses of a Fredholm scalar Toeplitz operator $T(g)$ can be effectively derived. Thus if the factorization index n of the function g is non-negative, then $T(g)$ is left–invertible and the operator $T(t^{-n})T^{-1}(g_0)$, where $g_0 := at^{-n}$, is one of the left–inverses for $T(g)$. On the other hand, if $n \leq 0$, then $T(g)$ is right–invertible. For the sake of convenience, in this paper the notation $T_r^{-1}(g)$ always means the operator $T^{-1}(a_0)T(t^{-n})$, which is one of right inverses for the operator $T(g)$. Besides, for $n > 0$ the kernel of the operator $T(t^{-n})$ is the linear span of the monomials $1, t, \cdots, t^{n-1}$, i.e. $\ker T(t^{-n}) = \operatorname{span}\{1, t, \cdots, t^{n-1}\}$. Moreover, if $T(g)$ is right–invertible and $\dim\ker T(g) = \infty$, then $T_r^{-1}(g)$ denotes one of right inverses of $T(g)$.

Assume now that $g \in L^\infty$ satisfies the matching condition (17) and the corresponding operator $T(g) : H^p \mapsto H^p$ is Fredholm. We already know that if $\operatorname{ind} T(g) > 0$, then

$$\ker T(g) = \operatorname{im} \mathbf{P}_\alpha^+(g) \dotplus \operatorname{im} \mathbf{P}_\alpha^-(g).$$

Now we want to derive an explicit description of the spaces $\operatorname{im} \mathbf{P}_\alpha^+(g)$ and $\operatorname{im} \mathbf{P}_\alpha^-(g)$, and the result below is the first step towards this goal.

Theorem 4.2 *Assume that $g \in L^\infty$ satisfies the matching condition $gg_\alpha = 1$ and the operator $T(g) : H^p \mapsto H^p$ is Fredholm. If* $\operatorname{ind} T(g) = n$, $n \in \mathbb{Z}$, *then g can be represented in the form*

$$g = \xi g_+ \chi^{-n} (g_+^{-1})_\alpha \,, \tag{30}$$

where g_+ and n occur in the Wiener–Hopf factorization

$$g = g_- t^{-n} g_+, \quad g_-(\infty) = 1, \tag{31}$$

of the function g, whereas $\xi \in \{-1, 1\}$ and is defined by

$$\xi = \left(\frac{\lambda}{\overline{\overline{\beta}}}\right)^n g_+^{-1}\left(\frac{1}{\overline{\overline{\beta}}}\right). \tag{32}$$

Proof. It is clear that g^{-1} possesses a weak Wiener–Hopf factorization

$$g^{-1} = g_-^{-1} t^n g_+^{-1}, \quad g_-^{-1}(\infty) = 1$$

in H^q. But $g^{-1} = g_\alpha$, so that

$$\begin{aligned} g_-^{-1} t^n g_+^{-1} &= (g_-)_\alpha\, t_\alpha^{-n}\, (g_+)_\alpha = (g_-)_\alpha\, (\alpha)^{-n}\, (g_+)_\alpha \\ &= (g_-)_\alpha\, \alpha_-^{-n}\, t^n\, \alpha_+^{-n}\, (g_+)_\alpha = \eta_- t^n \eta_+, \end{aligned} \tag{33}$$

where

$$\eta_+ = (g_-)_\alpha\, \alpha_+^{-n}, \quad \eta_- = (g_+)_\alpha\, \alpha_-^{-n}.$$

It follows from (5), (7) and from the properties of the factorization factors g_- and g_+ that $\eta_+ \in H^p$, $\eta_+^{-1} \in H^q$, $\eta_- \in \overline{H^q}$ and $\eta_-^{-1} \in \overline{H^p}$. Therefore, comparing representations (31) and (33), one obtains

$$g_-^{-1}\, \eta_-^{-1} = \eta_+ g_+ = \xi$$

where ξ is a complex number. Indeed, the product in the left-hand side belongs to $\overline{H^1}$ whereas the right-hand side is in H^1, and it is well-known that $\overline{H^1} \cap H^1 = \mathbb{C}$. Thus

$$g_-^{-1} = \xi\, \eta_-, \quad g_+ = \xi \eta_+^{-1},$$

so that

$$g = \xi^{-1} \eta_-^{-1} t^{-n} g_+ = \xi^{-1}\, (g_+)_\alpha^{-1}\, \alpha_+^{-n}\, t^{-n}\, g_+ = \xi^{-1} (g_+)_\alpha^{-1}\, \chi^{-n}\, g_+.$$

Using this identity one can represent g_α in the form $g_\alpha = \xi^{-1}\, g_+^{-1} \chi^n\, (g_+)_\alpha$, which leads to the equation $1 = g g_\alpha = (\xi^{-1})^2$, i.e $\xi = -1$ or $\xi = 1$. On the other hand, from $g_- = \xi^{-1} \eta_-^{-1}$ and $g_-(\infty) = 1$ we get

$$1 = \frac{1}{\xi} \lim_{t\to\infty} (g_+)_\alpha^{-1}(t) (\alpha_-(t))^n = \frac{1}{\xi}\, g_+^{-1}\left(\frac{1}{\overline{\overline{\beta}}}\right)\left(\frac{\lambda}{\overline{\overline{\beta}}}\right)^n,$$

and the relation (32) follows. ■

Remark 4.1 *Factorization* (30) *has been mentioned in [11] without proof but the factor ξ is missing there.*

Now we can introduce the following definition.

4.2 *The number ξ defined by the relation* (32) *is called the α-factorization signature, or simply, α-signature of g and is denoted by $\boldsymbol{\sigma}_\alpha(g)$.*

The α-signature plays an important role in the construction of bases of the kernels of generalized Toeplitz plus Hankel operator. In order to determine it, one has to evaluate the corresponding factorization factor at a point of the complex plane $\mathbb{C}$. In general situation, this is not an easy task at all. Nevertheless, for some classes of generating functions g, this specific characteristic can be easily found and such a possibility is discussed later on.

Theorem 4.3 *Let $g \in L^\infty$ be an α-matching function such that the operator $T(g) : H^p \to H^p$ is Fredholm and $n := \operatorname{ind} T(g) > 0$. If $g = g_- t^{-n} g_+$, $g_-(\infty) = 1$ is the corresponding Wiener–Hopf factorization of g in H^p, then the following systems of functions $\mathcal{B}^\pm_\alpha(g)$ form bases in the spaces $\operatorname{im} \mathbf{P}^\pm_\alpha(g)$:*

(i) If $n = 2m$, $m \in \mathbb{N}$, then

$$\mathcal{B}^\pm_\alpha(g) := \{g_+^{-1}(\chi^{m-k-1} \pm \boldsymbol{\sigma}_\alpha(g)\chi^{m+k}) : k = 0, 1, \cdots, m-1\},$$

and

$$\dim \operatorname{im} \mathbf{P}^\pm_\alpha(g) = m.$$

(ii) If $n = 2m+1$, $m \in \mathbb{Z}_+$, then

$$\mathcal{B}^\pm_\alpha(g) := \{g_+^{-1}(\chi^{m+k} \pm \boldsymbol{\sigma}_\alpha(g)\chi^{m-k}) : k = 0, 1, \cdots, m\} \setminus \{0\},$$

$$\dim \operatorname{im} \mathbf{P}^\pm_\alpha(g) = m + \frac{1 \pm \boldsymbol{\sigma}_\alpha(g)}{2}.$$

Remark 4.2 *If $n = 2m+1$, then the zero element belongs to one of the sets $\mathcal{B}^+_\alpha(g)$ or $\mathcal{B}^-_\alpha(g)$. Namely, for $k = 0$ one of the terms $\chi^m(1 \pm \boldsymbol{\sigma}_\alpha(g))$ is equal to zero.*

Proof. [Proof of Theorem 4.3] Observe that the restriction of the operators Pg_-I and $Pg_-^{-1}I$ on $\ker T(t^{-n}) = \operatorname{span}\{\chi^0, \chi, \cdots, \chi^{n-1}\}$ map $\ker T(t^{-n})$ into $\ker T(t^{-n})$, and on the space $\ker T(t^{-n})$ the above operators are inverses to each other.

Thus the elements $s_j = Pg_-\chi^j$, $j = 0, 1, \cdots, n-1$ belong to $\ker T(t^{-n})$ and

$$T^{-1}(g_0)s_j = g_+^{-1} P g_-^{-1} s_j = g_+^{-1}\chi^j,$$

where $g_0 := gt^n$. Here we used the relation $T^{-1}(g_0) = g_+^{-1}Pg_-^{-1}$ and the fact that $T^{-1}(g_0)s_j \in \ker T(g)$. Moreover, the set $\{T^{-1}(g_0)s_j : j = 0, \cdots, n-1\}$ constitutes a basis in $\ker T(g)$. Now one can use the factorization (29) and write

$$\begin{aligned}\mathbf{P}_\alpha(g)(T^{-1}(g_0)s_j) &= J_\alpha QgPT^{-1}(g_0)s_j \\ &= J_\alpha Q\boldsymbol{\sigma}_\alpha(g)(g_+^{-1})_\alpha \chi^{-n} g_+ P g_+^{-1}\chi^j = \boldsymbol{\sigma}_\alpha(g) g_+^{-1}\chi^{n-j-1},\end{aligned}$$

which leads to the representation

$$\mathbf{P}_\alpha^{\pm}(g)(T^{-1}(g_0)s_j) = \frac{1}{2}g_+^{-1}(\chi^j \pm \boldsymbol{\sigma}_\alpha(g)\chi^{n-j-1}), \quad j = 0, 1, , \cdots, n-1.$$

Assume now that $n = 2m$, $m \in \mathbb{N}$. If $j \in \{0, 1, \cdots, m-1\}$, it can be rewritten as $j = m-k-1$ with some $k \in \{0, 1, \cdots, m-1\}$ and vice versa. Hence

$$\chi^j \pm \boldsymbol{\sigma}_\alpha(g)\chi^{n-j-1} = \chi^{m-k-1} \pm \boldsymbol{\sigma}_\alpha(g)\chi^{m+k}, \quad j = 0, 1, \cdots, m-1.$$

On the other hand, if $j \geq m$, then $j = m+k$ for a $k \in \{0, 1, \cdots, m-1\}$, and

$$\chi^j \pm \boldsymbol{\sigma}_\alpha(g)\chi^{n-j-1} = \chi^{m+k} \pm \boldsymbol{\sigma}_\alpha(g)\chi^{m-k-1} = \pm\boldsymbol{\sigma}_\alpha(g)(\chi^{m-k-1} \pm \boldsymbol{\sigma}_\alpha(g)\chi^{m+k}).$$

Thus one obtains, maybe up to the factor -1, the same system of function that is derived for $j \in \{0, 1, \cdots, m-1\}$. So we conclude that if $n = 2m$, then

$$\dim \operatorname{im} \mathbf{P}_\alpha^{\pm}(g) = m,$$

and assertion (i) is shown.

The proof of assertion (ii) is similar to that of assertion (i). ∎

As was already mentioned, the evaluation of α-signature is a difficult problem. However, there are functions $g \in L^\infty$, the α-signature of which can be easily determined. Consider, for example, a function $g \in L^\infty$ continuous at one of the fixed points $t_\pm = (1 \pm \lambda)/\overline{\beta} \in \mathbb{T}$ of the mapping α. At any fixed point, such a function g can take only one of the two values, namely, $+1$ or -1, which leads to the following results.

Proposition 4.1 *Let $g \in L^\infty$ be an α-matching function such that*

(i) The operator $T(g) : H^p \to H^p$ is Fredholm with index n.

(ii) The function g is continuous at the point t_+ or t_-.

Then

$$\boldsymbol{\sigma}_\alpha(g) = g(t_+) \text{ or } \boldsymbol{\sigma}_\alpha(g) = g(t_-)(-1)^n.$$

Proof. Assume for definiteness that the function g is continuous at the point t_+. Condition (i) ensures that g admits a Wiener–Hopf factorization in H^p,

$$g = g_- t^{-n} g_+, \quad g(\infty) = 1,$$

and $g(t_+) \in \{-1, 1\}$. Now we approximate the function g as follows. Chose an $\varepsilon > 0$ and an open arc $\mathbb{T}_\varepsilon \subset \mathbb{T}$ such that

(i) The point t_+ belongs to the arc $\mathbb{T}_\varepsilon$.

(ii) $\alpha(\mathbb{T}_\varepsilon) = \mathbb{T}_\varepsilon$ that is that α is a homomorphism of $\mathbb{T}_\varepsilon$.

(iii) If g_ε denotes the function

$$g_\varepsilon(t) := \begin{cases} g(t_+) & \text{if } t \in \mathbb{T}_\varepsilon \\ g(t) & \text{otherwise} \end{cases},$$

then

$$||g - g_\varepsilon|| < \varepsilon.$$

It is clear that such an arc $\mathbb{T}_\varepsilon$ exists and that $g_\varepsilon (g_\varepsilon)_\alpha = 1$. If ε is small enough, then the operator $T(g_\varepsilon)$ is also Fredholm on H^p with the same index n. In this case, the function g_ε admits a Wiener–Hopf factorization in H^p,

$$g_\varepsilon = g_{\varepsilon,-} t^{-n} g_{\varepsilon,+}, \quad g_\varepsilon(\infty) = 1.$$

Since g_ε is Hölder continuous in a neighbourhood of the point t_+, Corollary 5.15 of [12] indicates that the functions $g_{\varepsilon,-}$ and $g_{\varepsilon,+}$ are also Hölder continuous in a neighbourhood of t_+. By Theorem 4.2, the function g_ε can be written in the form

$$g_\varepsilon = \boldsymbol{\sigma}_\alpha(g_\varepsilon) g_{\varepsilon,+} (g_{\varepsilon,+}^{-1})_\alpha \, \chi^{-n}. \tag{34}$$

But $g_\varepsilon(t_+) = \boldsymbol{\sigma}_\alpha(g_\varepsilon) g_{\varepsilon,+}(t_+) (g_{\varepsilon,+}^{-1})_\alpha(t_+) \, \chi^{-n}(t_+) = \boldsymbol{\sigma}_\alpha(g_\varepsilon)$ because $\alpha(t_+) = t_+$ and $\chi(t_+) = 1$. Moreover, we have

$$\boldsymbol{\sigma}_\alpha(g) = \left(\frac{\lambda}{\overline{\overline{\beta}}}\right)^n g_+^{-1}\left(\frac{1}{\overline{\overline{\beta}}}\right),$$

$$\boldsymbol{\sigma}_\alpha(g_\varepsilon) = \left(\frac{\lambda}{\overline{\overline{\beta}}}\right)^n g_{\varepsilon,+}^{-1}\left(\frac{1}{\overline{\overline{\beta}}}\right).$$

Our aim now is to show that if ε is small enough, then $\boldsymbol{\sigma}_\alpha(g_\varepsilon) = \boldsymbol{\sigma}_\alpha(g)$. Consider the functions

$$g t^n = g_- g_+, \quad g_\varepsilon t^n = g_{\varepsilon,-} g_{\varepsilon,+},$$

and note that the Toeplitz operators $T(gt^n)$ and $T(g_\varepsilon t^n)$ are invertible on H^p. Let us also assume that ε is chosen so small that

$$||T^{-1}(gt^n) - T^{-1}(g_\varepsilon t^n)|| < \left|\frac{\lambda}{\overline{\beta}}\right|^{-n} s_p^{-1}, \tag{35}$$

where s_p denotes the norm of the linear bounded functional $h \mapsto h(1/\overline{\beta})$, $h \in H^p$. Further, consider two uniquely solvable equations

$$T(gt^n)h = 1, \quad T(g_\varepsilon t^n)k = 1.$$

Using the above mentioned Wiener–Hopf factorization one obtains

$$\begin{aligned} h &= T^{-1}(gt^n)1 = g_+^{-1} P g_-^{-1} 1 = g_+^{-1}, \\ k &= T^{-1}(g_\varepsilon t^n)1 = g_{\varepsilon,+}^{-1} P g_{\varepsilon,-}^{-1} 1 = g_{\varepsilon,+}^{-1}, \end{aligned}$$

and

$$||g_+^{-1} - g_{\varepsilon,+}^{-1}|| \le ||T^{-1}(gt^n) - T^{-1}(g_\varepsilon t^n)||.$$

It follows that

$$\begin{aligned} |\boldsymbol{\sigma}_\alpha(g) - \boldsymbol{\sigma}_\alpha(g_\varepsilon)| &= \left| \left(\frac{\lambda}{\overline{\beta}}\right)^n \left(g_+^{-1}\left(\frac{1}{\overline{\beta}}\right) - g_{\varepsilon,+}^{-1}\left(\frac{1}{\overline{\beta}}\right)\right)\right| \\ &\le \left|\frac{\lambda}{\overline{\beta}}\right|^n s_p \, ||T^{-1}(gt^n) - T^{-1}(g_\varepsilon t^n)||, \end{aligned}$$

and relation (35) leads to the inequality

$$|\boldsymbol{\sigma}_\alpha(g) - \boldsymbol{\sigma}_\alpha(g_\varepsilon)| < 1.$$

Therefore,

$$\boldsymbol{\sigma}_\alpha(g) = \boldsymbol{\sigma}_\alpha(g_\varepsilon),$$

and we have

$$\boldsymbol{\sigma}_\alpha(g) = \boldsymbol{\sigma}_\alpha(g_\varepsilon) = g_\varepsilon(t_+) = g(t_+).$$

If g is continuous at the point t_-, the proof is analogous but one has take into account that $\chi(t_-) = -1$. ■

Remark 4.3 *In the proof we used the fact that for a fixed $\xi \in D^+$ the mapping $h \mapsto h(\xi)$ is a bounded linear functional on H^p. The boundedness of this functional can be easily seen. Indeed, by Caushy's integral formula for any polynomial P_n, one has*

$$P_n(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{P_n(e^{i\theta})e^{i\theta}}{e^{i\theta} - \xi}\, d\theta.$$

Using Hölder inequality, one obtains that on the set of all polynomials the linear functional $h \mapsto h(\xi)$ is bounded in the H^p-norm. Since this set is dense in H^p, our claim follows.

Note that the two cases above exhaust all the situations possible.

5 Bases of the kernels and cokernels of generalized Toeplitz plus Hankel operators.

In this section the structure of the kernel and cokernel of generalized Toeplitz plus Hankel operator $T(a) + H_\alpha(b)$ is described. The operators in question are studied under the condition that their generating functions $a, b \in L^\infty$ constitute a Fredholm matching pair (a, b). Recall that if a matching pair (a, b) is Fredholm, then it follows from (15) and (20) that $T(a)+H_\alpha(b)$ and $T(a)-H_\alpha(b)$ are Fredholm operators. Set $\kappa_1 := \operatorname{ind} T(c)$, $\kappa_2 := \operatorname{ind} T(d)$ and let $\mathbb{Z}_-$ refer to the set of all negative integers.

Theorem 5.1 *Assume that $(a, b) \in L^\infty \times L^\infty$ is a Fredholm matching pair. Then*

(i) If $(\kappa_1, \kappa_2) \in \mathbb{Z}_+ \times \mathbb{N}$, then the operators $T(a) + H_\alpha(b)$ and $T(a) - H_\alpha(b)$ are invertible from the right and

$$\ker(T(a) + H_\alpha(b)) = \operatorname{im} \mathbf{P}_\alpha^-(c) \dotplus \varphi_\alpha^+(\operatorname{im} \mathbf{P}_\alpha^+(d)),$$
$$\ker(T(a) - H_\alpha(b)) = \operatorname{im} \mathbf{P}_\alpha^+(c) \dotplus \varphi_\alpha^-(\operatorname{im} \mathbf{P}_\alpha^-(d)),$$

where the spaces $\operatorname{im} \mathbf{P}_c^\pm$ and $\operatorname{im} \mathbf{P}_d^\pm$ are described in Theorem 4.3, and the mappings $\varphi_\alpha^\pm$ are defined by (23).

(ii) If $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times (\mathbb{Z} \setminus \mathbb{N})$, then the operators $T(a) + H_\alpha(b)$ and $T(a) - H_\alpha(b)$ are invertible from the left and

$$\operatorname{coker}(T(a) + H_\alpha(b)) = \operatorname{im} \mathbf{P}_\alpha^-(\overline{d}) \dotplus \varphi_\alpha^+(\operatorname{im} \mathbf{P}_\alpha^+(\overline{c})),$$
$$\operatorname{coker}(T(a) - H_\alpha(b)) = \operatorname{im} \mathbf{P}_\alpha^+(\overline{d}) \dotplus \varphi_\alpha^-(\operatorname{im} \mathbf{P}_\alpha^-(\overline{c})),$$

and $\operatorname{im} \mathbf{P}_{\overline{d}}^\pm = \{0\}$ for $\kappa_2 = 0$.

(iii) If $(\kappa_1, \kappa_2) \in \mathbb{Z}_+ \times (\mathbb{Z} \setminus \mathbb{N})$, then

$$\ker(T(a) + H_\alpha(b)) = \operatorname{im} \mathbf{P}_\alpha^-(c), \quad \operatorname{coker}(T(a) + H_\alpha(b) = \operatorname{im} \mathbf{P}_\alpha^+(\overline{d}),$$
$$\ker(T(a) - H_\alpha(b)) = \operatorname{im} \mathbf{P}_\alpha^+(c) \quad \operatorname{coker}(T(a) - H_\alpha(b) = \operatorname{im} \mathbf{P}_\alpha^-(\overline{d}).$$

Proof. Note that all results concerning the kernels of the operators under consideration follow from Theorem 4.3 and representations (24). In order to describe the cokernels of the corresponding operators, let us recall that $\operatorname{coker}(T(a) \pm H_\alpha(b)) := \ker(T(a) \pm H_\alpha(b))^*$. Moreover, $(T(a) \pm H_\alpha(b))^* = T(\overline{a}) \pm H_\alpha(\overline{b}_\alpha)$ and the duo $(\overline{a}, \overline{b}_\alpha)$ is a matching pair with the subordinated pair $(\overline{d}, \overline{c})$. Further, if $g \in L^\infty$ and $\operatorname{Ind} c = r$, then $\operatorname{Ind} \overline{c} = -r$, and the description of the cokernel immediately follows from the previous results for the kernels of Toeplitz plus Hankel operators. ∎

It remains to consider the case $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{N}$. This situation is more involved and factorization (20) already indicates that for $\kappa_2 > 0$, the dimension of the kernel

diag $(T(a)+H_\alpha(b), T(a)-H_\alpha(b))$ may be smaller than κ_2. To treat this case, consider a number $n \in \mathbb{N}$ such that

$$0 \le 2n + \kappa_1 \le 1.$$

Such an n is uniquely defined and

$$2n + \kappa_1 = \begin{cases} 0, & \text{if } \kappa_1 \text{ is even}, \\ 1, & \text{if } \kappa_1 \text{ is odd}. \end{cases}$$

Now one can use the relation (25) and represent the operators $T(a) \pm H_\alpha(b)$ in the form

$$T(a) \pm H_\alpha(b) = (T(a\chi^{-n}) \pm H_\alpha(b\chi^n))T(\chi^n). \tag{36}$$

But $(a\chi^{-n}, b\chi^n)$ is a matching pair with the subordinated pair $(c\chi^{-2n}, d)$. Hence, the operators $T(a\chi^{-n}) \pm H_\alpha(b\chi^n)$ are subject to assertion (i) of Theorem 5.1. Thus they are right-invertible, and if κ_1 is even, then

$$\begin{aligned} \ker(T(a\chi^{-n}) + H_\alpha(b\chi^n)) &= \varphi_\alpha^+(\operatorname{im} \mathbf{P}_\alpha^+(d)), \\ \ker(T(a\chi^{-n}) - H_\alpha(b\chi^n)) &= \varphi_\alpha^-(\operatorname{im} \mathbf{P}_\alpha^-(d)), \end{aligned} \tag{37}$$

and if κ_1 is odd, then

$$\begin{aligned} \ker(T(a\chi^{-n}) + H_\alpha(b\chi^n)) &= \frac{1 - \boldsymbol{\sigma}_\alpha(c)}{2} c_+^{-1}\mathbb{C} \dotplus \varphi_\alpha^+(\operatorname{im} \mathbf{P}_\alpha^+(d)), \\ \ker(T(a\chi^{-n}) - H_\alpha(b\chi^n)) &= \frac{1 + \boldsymbol{\sigma}_\alpha(c)}{2} c_+^{-1}\mathbb{C} \dotplus \varphi_\alpha^-(\operatorname{im} \mathbf{P}_\alpha^-(d)), \end{aligned} \tag{38}$$

where the mappings $\varphi_\alpha^\pm$ depend on the functions $a\chi^{-n}$ and $b\chi^n$.

Theorem 5.2 *Let $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{N}$. Then*

(i) If κ_1 is odd, then

$$\ker(T(a) \pm H_\alpha(b)) = \\ T(\chi^{-n}) \left(\left\{ \frac{1 \mp \boldsymbol{\sigma}_\alpha(c)}{2} c_+^{-1}\mathbb{C} \dotplus \varphi_\alpha^\pm(\operatorname{im} \mathbf{P}_\alpha^\pm(d)) \right\} \cap \operatorname{im} T(\chi^n) \right);$$

(ii) If κ_1 is even, then

$$\ker(T(a) \pm H_\alpha(b)) = T(\chi^{-n}) \left(\left\{ \varphi_\alpha^\pm(\operatorname{im} \mathbf{P}_\alpha^\pm(d)) \right\} \cap \operatorname{im} T(\chi^n) \right),$$

and the mappings $\varphi_\alpha^\pm$ depend on the functions $a\chi^{-n}$ and $b\chi^n$.

Remark 5.1 *Using the fact that the system* $\{t^{k-1}\alpha_+^{-n}\}_{k=1}^n$ *forms a basis of* $\ker T^*(\chi^n) = \ker T(\overline{\chi}^n) = \ker T(t^{-n}\alpha_+^n)$*, it is easily seen that* $h \in H^p$ *belongs to* $\operatorname{im} T(\chi^n)$ *if and only if*

$$\widehat{(h\alpha_-^n)}_i = 0, \quad i = 0, 1, \cdots, n-1,$$

where $\widehat{(h\alpha_-^n)}_i$ *denotes the* i*-th Fourier coefficient of the function* $h\alpha_-^n$*.*

Proof. It follows immediately from representations (36)–(38). ∎

Theorem 5.2 can also be used to describe the cokernels of the operators $T(a) \pm H_\alpha(b)$ in the situation where $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{N}$. Indeed, recalling that $(T(a) \pm H_\alpha(b))^* = T(\overline{a}) \pm H_\alpha(\overline{b}_\alpha)$, and $(\overline{d}, \overline{c})$ is the subordinated pair for $(\overline{a}, \overline{b}_\alpha)$, one can note that the operators $T(\overline{d})$ and $T(\overline{c})$ are also Fredholm and

$$\operatorname{ind} T(\overline{d}) = -\kappa_2, \quad \operatorname{ind} T(\overline{c}) = -\kappa_1,$$

so $(-\kappa_2, \kappa_1) \in \mathbb{Z}_- \times \mathbb{N}$. Therefore, Theorem 5.2 applies and we can formulate the following result.

Theorem 5.3 *Let* $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{N}$*, and let* $m \in \mathbb{N}$ *satisfy the requirement*

$$1 \geq 2m - \kappa_2 \geq 0.$$

Then

(i) If κ_2 *is odd, then*

$$\operatorname{coker}(T(a) \pm H_\alpha(b)) = T(\chi^{-m}) \left(\left\{ \frac{1 \mp \boldsymbol{\sigma}_\alpha(\overline{d})}{2} \overline{d_-^{-1}} \mathbb{C} \dotplus \varphi_\alpha^\pm(\operatorname{im} \mathbf{P}_\alpha^\pm(\overline{c})) \right\} \cap \operatorname{im} T(\chi^m) \right).$$

(ii) If κ_2 *is even, then*

$$\ker(T(a) \pm H_\alpha(b)) = T(\chi^{-m}) \left(\left\{ \varphi_\alpha^\pm(\operatorname{im} \mathbf{P}_\alpha^\pm(\overline{c})) \right\} \cap \operatorname{im} T(\chi^m) \right),$$

and the mappings $\varphi_\alpha^\pm$ *depend on* $\overline{a}\chi^{-m}$ *and* $\overline{b}_\alpha \chi^m$*.*

Thus Theorems 5.1-5.3 offer an explicit description of the kernels and cokernels of the operators under consideration. On the other hand, the above approach can be also used to find generalized Toeplitz plus Hankel operators which are subject to Coburn–Simonenko theorem. Recall that some operators possessing this property have been already studied in Section 3.

Proposition 5.1 *Let $(a, b) \in L^\infty \times L^\infty$ be a matching pair with the subordinated pair (c, d), and let $T(c)$ be a Fredholm operator. Then:*

(i) If $\operatorname{ind} T(c) = 1$ *and* $\boldsymbol{\sigma}_\alpha(c) = 1$, *then* $\ker(T(a)+H_\alpha(b)) = \{0\}$ *or* $\operatorname{coker}(T(a)+H_\alpha(b)) = \{0\}$.

(ii) If $\operatorname{ind} T(c) = -1$ *and* $\boldsymbol{\sigma}_\alpha(c) = 1$, *then* $\ker(T(a)-H_\alpha(b)) = \{0\}$ *or* $\operatorname{coker}(T(a)-H_\alpha(b)) = \{0\}$.

(iii) If $\operatorname{ind} T(c) = 0$, *then* $\ker(T(a) \pm H_\alpha(b)) = \{0\}$ *or* $\operatorname{coker}(T(a) \pm H_\alpha(b)) = \{0\}$.

Proof. The proof of the last theorem is similar to the proof of Theorem 3.1, but in the proofs of assertions (i) and (ii) one, respectively, has to use the fact that $c_+^{-1} \in \ker(T(a) + H_\alpha(b))$ and $c_+^{-1} \in \ker(T(a\chi^{-1}) - H_\alpha(b\chi))$. These inclusions can be verified by straightforward computations. Thus consider, for example, the expression $(T(a\chi^{-1}) - H_\alpha(b\chi))c_+^{-1}$. Taking into account factorization (30) and using the relations $c_- = (c_+^{-1})_\alpha$ and $a = bc$, one obtains

$$\begin{aligned}(T(a\chi^{-1}) - H_\alpha(b\chi))c_+^{-1} &= Pbc\chi^{-1}c_+^{-1} - Pb\chi QJ_\alpha c_+^{-1} \\ &= Pbc_+\chi c_-\chi^{-1}c_+^{-1} - Pb\chi(c_+^{-1})_\alpha\chi^{-1} = 0,\end{aligned}$$

and the inclusion $c_+^{-1} \in \ker(T(a\chi^{-1}) - H_\alpha(b\chi)$ is proved. ∎

Remark 5.2 *Let us emphasize that Proposition 5.1 is valid without any assumption about Fredlomness, semi-Fredholness, or even normal solvability of the operators $T(a) \pm H_\alpha(b)$. However, if one of these operators is Fredholm and its index is known, then Proposition 5.1 allows one to compute the kernel and cokernel dimension of the operator under consideration. For the case of piecewise continuous generating function see also Section 6.*

Corollary 5.1 *Let $b \in L^\infty$ be a matching function. If $T(b_\alpha)$ is a Fredholm operator, then:*

(i) If $\operatorname{ind} T(b_\alpha) = 1$, *and* $\boldsymbol{\sigma}_\alpha(b_\alpha) = 1$, *then* $\ker(I + H_\alpha(b))$ *or* $\operatorname{coker}(I + H_\alpha(b))$ *is trivial.*

(ii) If $\operatorname{ind} T(b_\alpha) = -1$, *and* $\boldsymbol{\sigma}_\alpha(b_\alpha) = 1$, *then* $\ker(I - H_\alpha(b))$ *or* $\operatorname{coker}(I - H_\alpha(b))$ *is trivial.*

(iii) If $\operatorname{ind} T(b_\alpha) = 0$, *then* $\ker(I \pm H_\alpha(b))$ *or* $\operatorname{coker}(I \pm H_\alpha(b))$ *is trivial.*

These results are direct consequences of Proposition 5.1, since if b is a matching function, then $(1, b)$ is a matching pair with the subordinate pair (b_α, b).

In conclusion of this section, we would like to mention that in certain cases the condition of Fredholmness of the operator $T(d)$ can be dropped. However, we are not going to pursue this matter here.

6 Fredholmness of generalized Toeplitz plus Hankel operators with piecewise continuous generating functions

Assume α is the Carleman shift (4) changing the orientation, i.e. $|\beta| > 1$. As was already mentioned, the mapping $\alpha : \mathbb{T} \to \mathbb{T}$ has two fixed points, namely

$$t_{\pm} = \frac{1 \pm i\sqrt{|\beta|^2 - 1}}{\overline{\beta}}.$$

By $\mathbb{T}_\alpha^+$ and $\mathbb{T}_\alpha^-$ we denote the closed arcs of $\mathbb{T}$ which, respectively, join t_+ with t_- and t_- with t_+ and inherit the orientation of $\mathbb{T}$. Further, let us introduce the functions

$$\nu_p(y) := \frac{1}{2}\left(1 + \coth\left(\pi\left(y + \frac{i}{p}\right)\right)\right), \quad h_p(y) := \sinh^{-1}\left(\pi\left(y + \frac{i}{p}\right)\right),$$

where $y \in \overline{\mathbb{R}}$, and $\overline{\mathbb{R}}$ refers to the two-point compactification of $\mathbb{R}$. Set $\overset{\circ}{\mathbb{T}}{}_\alpha^+ := \mathbb{T}_\alpha^+ \setminus \{t_+, t_-\}$.

Theorem 6.1 *If $a, b \in PC$, then the operator $T(a) + H_\alpha(b)$ is Fredholm if and only if the matrix*

$$\operatorname{smb}(T(a) + H_\alpha(b))(t, y) :=$$
$$\begin{pmatrix} a(t+0)\nu_p(y) + a(t-0)(1-\nu_p(y)) & \dfrac{b(t+0) - b(t-0)}{2i}\, h_p(y) \\ \dfrac{b(\alpha(t)-0) - b(\alpha(t)+0)}{2i}\, h_p(y) & a(\alpha(t)+0)\nu_p(y) + a(\alpha(t)-0)(1-\nu_p(y)) \end{pmatrix}$$

is invertible for every $(t, y) \in \overset{\circ}{\mathbb{T}}{}_\alpha^+ \times \overline{\mathbb{R}}$ and the function

$$\begin{aligned} \operatorname{smb}(T(a) + H_\alpha(b))(t, y) := {} & a(t+0)\nu_p(y) + a(t-0)(1-\nu_p(y)) \\ & + \mu(t)\, \frac{b(t+0) - b(t-0)}{2}\, h_p(y) \end{aligned}$$

where

$$\mu(t) = \begin{cases} 1 & \textit{if} \quad t = t_+ \\ -1 & \textit{if} \quad t = t_- \end{cases},$$

does not vanish on $\{t_+, t_-\} \times \overline{\mathbb{R}}$.

Note that although this result cannot be found in the literature, it is not entirely new. It can be proved similarly to [14] by using localization technique and the two-projection theorem. For classical Toeplitz plus Hankel operators an analogous result

is presented in [13]. On the other hand, an index formula can be established following ideas of [15]. Moreover, if the generating functions constitute a Fredholm matching pair (a, b), the kernel and cokernel of the operator $T(a) + H_\alpha(b)$ can be described using results of Section 4. For non-Fredholm matching pairs (a, b) the situation is more complicated. However, if a and b are piecewise continuous functions, this case is still treatable. For example, the following result is true.

Theorem 6.2 *Let $a, b \in PC$ and (a, b) be a matching pair. If the operator $T(a) + H_\alpha(b) : H^p \to H^p$ is Fredholm, then there is an interval (p, p_0), $p < p_0$ such that for all $r \in (p, p_0)$ the pair (a, b) and both operators $T(a) \pm H_\alpha(b) : H^r \to H^r$ are Fredholm,*

$$\begin{aligned} \ker(T(a) + H_\alpha(b))\,|_{H^r \to H^r} &= \ker(T(a) + H_\alpha(b))\,|_{H^p \to H^p}\,, \\ \operatorname{coker}(T(a) + H_\alpha(b))\,|_{H^r \to H^r} &= \operatorname{coker}(T(a) + H_\alpha(b))\,|_{H^p \to H^p}\,, \end{aligned}$$

and the kernel and cokernel of the operator $T(a) + H_\alpha(b) : H^r \to H^r$ are described by Theorems 5.1–5.2, 5.3.

The proof is similar to the corresponding considerations of [8].

7 How to determine the α-factorization signature for piecewise continuous functions

Suppose that a function $g \in PC$ satisfies the following two conditions.

(i) $g g_\alpha = 1$.

(ii) The operator $T(g)$ is Fredholm on H^p.

In order to determine $\boldsymbol{\sigma}_\alpha(g)$ we need a special factorization of the function g. More precisely, this function has to be represented as

$$g = \psi_{\beta_+, t_+} g_+, \ \text{ or } \ g = \psi_{\beta_-, t_-} g_-, \tag{39}$$

where the function $\psi_{\beta_\pm, t_\pm}$ and $g_\pm$ possess the following properties.

(i) The function ψ_{β_+, t_+} has a jump at the point t_+ and is continuous on the arc $\mathbb{T} \setminus \{t_+\}$.

(ii) The function ψ_{β_-, t_-} has a jump at the point t_- and is continuous on the arc $\mathbb{T} \setminus \{t_-\}$.

(iii) $\psi_{\beta_+, t_+}(\psi_{\beta_+, t_+})_\alpha = 1, \quad \psi_{\beta_-, t_-}(\psi_{\beta_-, t_-})_\alpha = 1.$

(iv) The function g_+ and g_- are continuous at the points t_+ and t_-, respectively.

It is clear that if such factorizations take place, then $g_+, g_- \in PC$ and

$$g_+(g_+)_\alpha = 1, \quad g_-(g_-)_\alpha = 1$$

It turns out that all the factorizations mentioned do really exist, and below we show how to construct them.

Let $z \neq 0$ be a complex number, and let $\arg z$ stand for that value of the argument of z, which is located in the interval $(-\pi, \pi]$. For $\beta \in \mathbb{C}$, $\operatorname{Re}\beta \in (-1/q, 1/p)$, and $\tau \in \mathbb{T}$ consider the function $\varphi_{\beta,\tau}(t) \in PC$ defined by

$$\varphi_{\beta,\tau}(t) := \exp\{i\beta \arg(-t/\tau)\}, \quad t \in \mathbb{T}. \tag{40}$$

It is easily seen that $\varphi_{\beta,\tau}$ has at most one discontinuity, namely, a jump at the point τ and

$$\varphi_{\beta,\tau}(\tau+0) = \exp\{-i\pi\beta\}, \quad \varphi_{\beta,\tau}(\tau-0) = \exp\{i\pi\beta\}.$$

Recall a useful factorization of the function $\varphi_{\beta,\tau}$, viz.

$$\varphi_{\beta,\tau}(t) = \xi_{-\beta}(t)\eta_\beta(t), \tag{41}$$

where

$$\xi_\beta(t) = \left(1 - \frac{\tau}{t}\right)^\beta := \exp\left\{\beta \log\left|1 - \frac{\tau}{t}\right| + i\beta \arg\left(1 - \frac{\tau}{t}\right)\right\},$$
$$\eta_\beta(t) = \left(1 - \frac{t}{\tau}\right)^\beta := \exp\left\{\beta \log\left|1 - \frac{t}{\tau}\right| + i\beta \arg\left(1 - \frac{t}{\tau}\right)\right\}.$$

Note that the representation (41) is a Wiener-Hopf factorization with the factorization index zero. Therefore, the Toeplitz operator with generating function (40) is invertible on the space H^p (see [4, Sections 5.35 and 5.36]).

Let t_+ and T_- be the fixed points of the mapping α. Recall that $t_+ = (1+\lambda)/\overline{\beta}$, $t_- = (1-\lambda)/\overline{\beta}$, where $\lambda := i\sqrt{|\beta|^2 - 1}$. According to [4, Sections 5.35 and 5.36], a function $g \in PC$, such that $T(g)$ is Fredholm, can be written in one of the following form

$$g = \varphi_{\beta_+,t_+} g_1, \quad g = \varphi_{\beta_-,t_-} g_2 \tag{42}$$

where $\operatorname{Re}\beta_{\pm,t_\pm} \in (-1/q, 1/p)$ and the functions $g_1, g_2 \in PC$ are continuous at the points t_+ and t_-, respectively. However, the representations (42) are of no use in the present situation since the products $\varphi_{\beta_{\pm,t_\pm}}(\varphi_{\beta_{\pm,t_\pm}})_\alpha$ are not equal to 1. Inspired by representation (41) and by Theorem 4.2 we define functions ψ_{β_+,t_+} and ψ_{β_-,t_-} by

$$\psi_{\beta_+,t_+}(t) := \eta_{\beta_+,t_+}(t)\eta_{-\beta_+,t_+}(\alpha(t)), \quad t \neq t_+\,, \tag{43}$$
$$\psi_{\beta_-,t_-}(t) := \eta_{\beta_-,t_-}(t)\eta_{-\beta_-,t_-}(\alpha(t)), \quad t \neq t_-\,. \tag{44}$$

We are going to study properties of the functions (43) and (44). Let us deal with the function (43). The other one can be treated analogously. First of all, we note that

$$\eta_{\beta_+,t_+} \in H^q, \quad \eta_{\beta_+,t_+}^{-1} \in H^p, \quad (\eta_{-\beta_+,t_+})_\alpha \in \overline{H^p}, \quad (\eta_{-\beta_+,t_+})_\alpha^{-1} \in \overline{H^q},$$

where $1/p+1/q=1$. Thus if one shows that ψ_{β_+,t_+} is a piecewise continuous function, then representation (43) is a weak Wiener–Hopf factorization for this function.

Due to our agreement about the choice of $\arg z$, $z \neq 0$, one has

$$\arg \frac{t-t_+}{\alpha(t)-t_+} + 2k(t)\pi = \arg\left(1 - \frac{t}{t_+}\right) - \arg\left(1 - \frac{\alpha(t)}{t_+}\right),$$

where $k(t)$ is a uniquely determined integer, $t \neq t_+$. Now we note that

$$\alpha(t) - t_+ = \alpha(t) - \alpha(t_+) = \frac{(|\beta|^2-1)(t-t_+)}{(\overline{\beta}t-1)(\overline{\beta}t_+-1)}.$$

Therefore,

$$\frac{t-t_+}{\alpha(t)-t_+} = \frac{(\overline{\beta}t-1)i\sqrt{|\beta|^2-1}}{|\beta|^2-1}, \quad t \neq t_+, \tag{45}$$

and

$$\lim_{t\to t_+} \frac{t-t_+}{\alpha(t)-t_+} = -1.$$

Consider now the argument of the function (45). It is equal to $\arg(i(\overline{\beta}t-1))$, and we will show that

$$\lim_{t\to t_+\pm 0} \arg(i(\overline{\beta}t-1)) = \mp\pi.$$

In order to study $\arg(i(\overline{\beta}t-1))$ represent function $t \mapsto i(\overline{\beta}t-1)$ as

$$i(\overline{\beta}t-1) = i(\overline{\beta}t_+\exp(-i\theta)-1), \quad \theta \in (0,2\pi).$$

Using the relation $\exp(-i\theta) = \cos\theta - i\sin\theta$, one can rewrite it in the form

$$i(\overline{\beta}t-1) = \sqrt{|\beta|^2-1}\,(\sin\theta-\cos\theta) + i(\cos\theta+\sin\theta-1). \tag{46}$$

Consequently,

$$\lim_{t\to t_+-0} \arg\left(\frac{t-t_+}{\alpha(t)-t_+}\right) = \lim_{\theta\to 0}\arg(i(\overline{\beta}t-1))$$
$$= \lim_{\theta\to 0}\arg(\sqrt{|\beta|^2-1}\,(\sin\theta-\cos\theta)+i(\cos\theta+\sin\theta-1)) = \pi,$$

because $\cos\theta+\sin\theta-1>0$ for θ small enough.

Analogously,

$$\lim_{t\to t_+ +0} \arg\left(\frac{t-t_+}{\alpha(t)-t_+}\right) = -\pi.$$

Indeed, using the relation (46) once again, we obtain

$$\begin{aligned}\lim_{t\to t_+ +0} \arg\left(\frac{t-t_+}{\alpha(t)-t_+}\right) &= \lim_{\theta\to 2\pi} \arg(\sqrt{|\beta|^2-1}\,(\sin\theta-\cos\theta) \\ &\quad + i(\cos\theta+\sin\theta-1)) = -\pi.\end{aligned}$$

It is clear that the function

$$t \to \arg\left(\frac{t-t_+}{\alpha(t)-t_+}\right)$$

is continuous on $\mathbb{T}\setminus\{t_+\}$. Hence, it is piecewise continuous on $\mathbb{T}$, having only one jump at the point t_+ if $\beta \neq 0$. Moreover, since both functions ψ_{β_+,t_+} and

$$\exp\left\{\beta_+ \log\left|\frac{t-t_+}{\alpha(t)-t_+}\right| + i\beta_+ \arg\left(\frac{t-t_+}{\alpha(t)-t_+}\right)\right\},$$

are continuous on $\mathbb{T}\setminus\{t_+\}$ and do not vanish there, we obtain that $t \to k(t)$, $t \neq t_+$ is also continuous on $\mathbb{T}\setminus\{t_+\}$, hence it is even constant. Now let us choose $t_0 \in \mathbb{T}$ such that $\alpha(t_0) = t_+$. Then

$$\arg\left(1-\frac{\alpha(t_0)}{t_+}\right) = 0, \quad \arg\left(1-\frac{t_0}{t_+}\right) \in (-\pi,\pi],$$

so $k(t) = 0$.

Thus it is shown that

$$\arg\left(1-\frac{t}{t_+}\right) - \arg\left(1-\frac{\alpha(t)}{t_+}\right) = \arg\frac{t-t_+}{\alpha(t)-t_+}, \quad t \neq t_+.$$

As a consequence, the function ψ_{β_+,t_+} possesses the following properties.

(i) The function ψ_{β_+,t_+} has a jump at the point t_+,

$$\lim_{t\to t_+ -0} \psi_{\beta_+,t_+}(t) = \exp(i\pi\beta_+),$$
$$\lim_{t\to t_+ +0} \psi_{\beta_+,t_+}(t) = \exp(-i\pi\beta_+),$$

and it is continuous on $\mathbb{T}\setminus\{t_+\}$.

(ii) The function ψ_{β_+,t_+} satisfies the relation

$$\psi_{\beta_+,t_+}(\psi_{\beta_+,t_+})_\alpha = 1.$$

Now let us show that the Toeplitz operator $T(\psi_{\beta_+,t_+})$ is invertible in H^p. Indeed, one has

$$\psi_{\beta_+,t_+} = \varphi_{\beta_+,t_+} h, \quad h = \frac{\psi_{\beta_+,t_+}}{\varphi_{\beta_+,t_+}},$$

where φ_{β_+,t_+} is defined by (40) and h is a continuous function which does not vanish on $\mathbb{T}$. Then the operator $T(\psi_{\beta_+,t_+}) - T(\varphi_{\beta_+,t_+})T(h)$ is compact. Since $T(\varphi_{\beta_+,t_+})T(h)$ is a Fredholm operator, the operator $T(\psi_{\beta_+,t_+})$ is also Fredholm, and the factorization (43) is in fact a Wiener–Hopf factorization with the factorization index 0. Hence, the operator $T(\psi_{\beta_+,t_+})$ is invertible in H^p.

Note that the factorization (43) is not normalized, that is $\eta_{-\beta_+,t_+}$ is not equal to 1 at infinity. However, the normalization can be easily achieved by multiplying the factor $(1-\alpha(t)/t_+)^{\beta_+}$ by the number $c := (1-(\overline{\beta})^{-1}/t_+)^{-\beta_+}$ and the factor $(1-\alpha(t)/t_+)^{-\beta_+}$ by the number c^{-1}.

Thus the factorization signature $\boldsymbol{\sigma}(\psi_{\beta_+,t_+})$ of the function ψ_{β_+,t_+} is equal to 1. This can be also obtained by observing that

$$\psi_{\beta_+,t_+}(t_-) = 1.$$

Theorem 7.1 *Let $g_1 \in PC$ be as above. Then*

$$\boldsymbol{\sigma}_\alpha(g) = \boldsymbol{\sigma}_\alpha(g_+) = \boldsymbol{\sigma}_\alpha(g_-).$$

The proof of these results run in parallel to the corresponding results of [8, Section 8].

Thus if the generating functions a and b of the generalized Toeplitz plus Hankel operator $T(a) + H_\alpha(b)$ are piecewise continuous and satisfy the matching condition (17), then representations (39) allows one to find α-signature of the corresponding auxiliary functions $c, \overline{c}$ and $d, \overline{d}$ and, consequently, to obtain an effective and complete description of the kernel and cokernel of the operator under consideration.

Remark 7.1 *If $a \in L^\infty$ is a matching function having one-sided limits at the points $t = t_\pm$, then Theorem 7.1 is still true. Moreover, the α-factorization signature can be effectively calculated even in the case where a has one-sided limits only at the one of the fixed points t_+ or t_-.*

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